AN IDENTITY FOR A CLASS OF ARITHMETICAL FUNCTIONS OF SEVERAL VARIABLES

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ABSTRACT. Johnson [1] evaluated the sum \( \sum_{d \mid n} |C(d;r)| \), where \( C(n;r) \) denotes Ramanujan's trigonometric sum. This evaluation has been generalized to a wide class of arithmetical functions of two variables. In this paper, we generalize this evaluation to a wide class of arithmetical functions of several variables and deduce as special cases the previous evaluations.

KEY WORDS AND PHRASES. Arithmetical functions of several variables, multiplicative functions, Ramanujan's sum and its generalizations.

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1. INTRODUCTION.

In [1], Johnson evaluated the sum

\[
\sum_{d \mid n} |C(d;r)|,
\]

where \( C(n;r) \) denotes Ramanujan's trigonometric sum. This evaluation has been generalized by Chidambaraswamy and Krishnaiah [2], Johnson [3], and Redmond [4]. The generalization given by Chidambaraswamy and Krishnaiah is the most extensive one and contains the other evaluations as special cases. They evaluated the sum

\[
\sum_{d^k \mid n} |S^{(k)}(d^k;r)|,
\]

where \( k \) is a positive integer and

\[
S^{(k)}(n;r) = S^{(k)}_{g,h}(n;r) = \sum_{d^k \mid (n,r^k)} g(d)\mu(r/d)h(r/d),
\]

g and \( h \) being given arithmetical functions, \( \mu \) being the well-known Möbius function and \( (x,y)_k \) standing for the greatest common \( k \)th power divisor of \( x \) and \( y \).

In this paper, we shall evaluate the more extensive sum

\[
\sum_{d_1^k \mid n_1} \cdots \sum_{d_j^k \mid n_j} |S^{(k)}(d_1^k,\ldots,d_j^k,n_{j+1},\ldots,n_k;r)|.
\]
where
\[ S^{(k)}(n_1, \ldots, n_u; r) = \sum_{d^k \mid (n_1, \ldots, n_u)} g(d)\mu(r/d)h(r/d). \]

Here \((n) = (n_1, \ldots, n_u)\), the greatest common divisor of \(n_1, \ldots, n_u\).

2. RESULTS.

For a positive integer \(k\) let \(\tau_k\) denote the arithmetical function such that \(\tau_k(n)\) is the number of positive \(k\)th power divisors of \(n\).

For a given \((u + 1)\)-tuple \(n_1, \ldots, n_u, r\) of positive integers let \(\hat{r}\) denote the largest divisor of \(r\) such that \((\hat{r}, n_i) = 1\) for all \(i = 1, \ldots, u\). Also for each \(i = 1, \ldots, u\) let \(\hat{n}_i\) denote the largest divisor of \(n_i\) such that \((\hat{n}_i, r) = 1\). We write \(\hat{r}\) for \(r/\hat{r}\) and \(\hat{n}_i\) for \(n_i/\hat{n}_i\). The symbol \(r_\ast\) denotes the quotient of \(r\) by its largest squarefree divisor.

Let \(n_i = \prod p_i^{\alpha_i}(a_i = a_i(p))\), \(r = \prod p_i^{\beta_i}(b = b(p))\) be the canonical decompositions of \(n_i (i = 1, \ldots, u)\) and \(r\). When \(r^k \mid n_i\), let \(c_i = c_i(p, k)\) be determined so that \(p^k \mid n_i/r^k\) and \(p^k(c_i + 1) + n_i/r^k\); that is, \(c_i = [a_i/k] - b + 1\) if \(b \geq 1\), and \(c_i = [a_i/k]\) if \(b = 0\).

THEOREM. If \(g\) is a completely multiplicative function, \(h\) a multiplicative function and \(1 \leq j \leq u\), then
\[
\sum_{d^k \mid n_1} \cdots \sum_{d^k \mid n_j} |S^{(k)}(d^k_1, \ldots, d^k_j, n^k_{j+1}, \ldots, n^k_u; r)|
= \tau_k(\hat{n}_1) \cdots \tau_k(\hat{n}_j) |g(\hat{r}_\ast)|
\times \prod_{p \mid \hat{r}} \left(\{((c_1+1) \cdots (c_j+1) - c_1 \cdots c_j) | h(p)\} + c_1 \cdots c_j | g(p) - h(p)\} \right)
\times \prod_{p \mid \hat{r}} ((c_1+1) \cdots (c_j+1) | h(p)|
\tag{2.1}
\]

or 0 according as \(r^k \mid (n_1, \ldots, n_j, n^k_{j+1}, \ldots, n^k_u)\) or not, where \(a = \min(a_1, \ldots, a_u)\). (If \(j = u\), we put \(a = \infty\)).

PROOF. Let \(r^k \mid (n_1, \ldots, n_j, n^k_{j+1}, \ldots, n^k_u)\). Suppose \(d^k \mid n_i\) for each \(i = 1, \ldots, j\). Write
\[
S^{(k)}(d^k_1, \ldots, d^k_j, n^k_{j+1}, \ldots, n^k_u; r) = \sum_{\delta \mid r} g(\delta)\mu(\delta/r)h(\delta/r)
\]
Here \(r_\ast\) \((d_1, \ldots, d_j, n_{j+1}, \ldots, n_u)\) and so \(\mu(\delta/r) = 0\) for all \(\delta\) in the sum. Thus the left-hand side of (2.1) is equal to 0.

Let \(r^k \mid (n_1, \ldots, n_j, n^k_{j+1}, \ldots, n^k_u)\). Suppose \(d^k \mid n_i\) for each \(i = 1, \ldots, j\). Let \(\hat{d}_i\) and \(\hat{d}_i\) be defined in a similar way to \(\hat{n}_i\) and \(\hat{n}_i\). Then the multiplicativity of \(S^{(k)}(n_1, \ldots, n_u; r)\) in the variables \(n_1, \ldots, n_u, r\) implies
\[
S^{(k)}(d^k_1, \ldots, d^k_j, n^k_{j+1}, \ldots, n^k_u; r) = S^{(k)}(d^k_1, \ldots, d^k_j, d^k_{j+1}, \ldots, d^k_u; \hat{r}_\ast)S^{(k)}(d^k_{j+1}, \ldots, d^k_u; \hat{r}_\ast)S^{(k)}(d^k_{j+1}, \ldots, d^k_u; \hat{r}_\ast)\]
\[
= S^{(k)}(d^k_1, \ldots, d^k_j, \hat{d}_{j+1}, \ldots, \hat{d}_u; \hat{r}_\ast)\mu(\hat{r}_\ast)h(\hat{r}_\ast).
\]
Thus, denoting by \( L \) the left-hand side of (1.1), we obtain

\[
L = \left| h(\mathcal{F}) \right| \sum_{d_1^k \mid n_1} \cdots \sum_{d_j^k \mid n_j} \left| S^{(k)}(d_1^k, \ldots, d_j^k, n_1^k, \ldots, n_j^k) \right|
\]

Thus the sum over \( e_1, \ldots, e_j \) is equal to \( r_k(\tilde{a}_1) \cdots r_k(\tilde{a}_j) \).

By the multiplicativity of the function \( S^{(k)}(n_1, \ldots, n_j) \) and the properties of the Möbius function \( \mu \), we have

\[
\sum_{d_1^k \mid n_1} \cdots \sum_{d_j^k \mid n_j} \left| S^{(k)}(d_1^k, \ldots, d_j^k, n_1^k, \ldots, n_j^k) \right| = \prod_{p \mid \mathcal{F}} \sum_{i_1 = 0}^{[a_1/k]} \cdots \sum_{i_j = 0}^{[a_j/k]} \left| S^{(k)}(p^{i_1} \cdots p^{i_j}, p^{a_j + 1} \cdots p^{a_j}) \right|
\]

Thus

\[
L = r_k(\tilde{a}_1) \cdots r_k(\tilde{a}_j) \left| g(r_u) \right| h(\mathcal{F}) \| \times \prod_{p \mid \mathcal{F}} \sum_{b \leq a} \left\{ \left( (c_1 + 1) \cdots (c_j + 1) - c_1 \cdots c_j \right) \left| g(p^b - 1)h(p) \right| + c_1 \cdots c_j \left| g(p^b - 1) - g(p) - h(p) \right| \right\}
\]

If \( p \mid \mathcal{F} \), then \( b = 1 \) and \( c_1 = \cdots = c_j = a = 0 \). We thus arrive at our result.

**EXAMPLES.** If \( j = u = 1 \) in the Theorem, we obtain the result given in [2]; that is,

\[
\sum_{d_1^k \mid n_1} \left| S^{(k)}(d_1^k, r_1^k) \right| = r_k(\tilde{a}_1) \left| g(r_u) \right| \prod_{p \mid \mathcal{F}} \left( \left| h(p) \right| + c_1 \left| g(p) - h(p) \right| \right)
\]

or 0 according as \( r_k^k \mid n_1 \) or not. For special cases of (2.2) we refer to [2]. If \( g(n) = n^n \) and \( h(n) = 1 \) for all \( n \in \mathbb{N} \), then the function \( S^{(k)}(n_1, \ldots, n_j) \) reduces to the generalized Ramanujan's sum given in [5]. If in addition, \( k = 1 \), then we obtain the generalized Ramanujan's sum given in [6]. Thus the Theorem could be specialized to those functions, too.
REFERENCES


